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Reanalysis of the Das et al. sum rule and application to chiral $O(p^4)$ parameters

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Abstract. A sum rule due to Das et al. is reanalyzed using a euclidian space approach and a Padé resummation procedure. It is shown that the result is essentially determined by the matrix elements of dimension six and dimension eight operators which have recently been measured by the ALEPH collaboration. The result is further improved by using the vector spectral function which must be extrapolated to the chiral limit. This extrapolation is shown to be reliably performed under the constraint of a set of sum rules. The sum rule is employed not as an approximation to $M_{\pi^+} - M_{\pi^0}$ but as an exact result for a chiral low-energy parameter. A sufficiently precise evaluation provides also an estimate for a combination of subleading electromagnetic low-energy parameters.

1 Introduction

Chiral perturbation theory (e.g. [1] for a comprehensive review) is now claiming to reach such a high degree of accuracy in some situations that it is becoming necessary to deal quantitatively with radiative corrections in low energy processes. An important example is the pion-pion scattering amplitude for which the two-loop contribution has recently been evaluated [2,3]. The relevance of this reaction for probing experimentally a basic issue in the spontaneous breaking of chiral symmetery in QCD is discussed in some detail in [2]. Calculations of radiative corrections have started to be performed both for the pionium atomic bound state (e.g. [4] and references therein), in view of an experiment planning to form pionium atoms at CERN [5], and for the scattering amplitude [6,7]. The framework for performing such calculations is a natural extension of the conventional chiral expansion to include the photon as a dynamical quantum field [8]. This extension brings in a set of new, a priori unknown, low-energy constants. At chiral order two, a single constant appears (which will be denoted by C below) while at the next chiral order, one has to deal with fourteen new constants called k_i in the case of the $SU(2) \times SU(2)$ chiral group [6,7].

The purpose of this paper is to reanalyze the classic sum rule of Das et al. [9]. Since experimental data has started to become available from τ decays into hadrons, the sum rule was discussed several times in the literature [13–15] using as input experimentally measured vector and axial-vector spectral functions. One must be cautious, however, that the sum rule can at best provide an approximation to the $\pi^+ - \pi^0$ mass difference if the spectral functions are not extrapolated to the chiral limit. Indeed, the derivation is made in the limit, $m_u = m_d = 0$ and, strictly speaking, the integral diverges if one uses physical spectral functions over an infinite range. In modern context, the sum rule must be interpreted as an *exact* result for the low-energy constant C. This constant appears in the leading term of the chiral expansion of the $\pi^+ - \pi^0$ mass difference

$$M_{\pi^+}^2 - M_{\pi^0}^2 = \frac{2e^2C}{F^2} + O(e^2M_{\pi^0}^2) + O((m_u - m_d)^2) + O(e^4) \ . \qquad (1)$$

The corrective terms $O(e^2M_{\pi^0}^2)$ and $O((m_u - m_d)^2)$ involve a number of low energy constants k_i and one $O(p^4)$ constant (l_7) respectively [7,10]. The order of magnitude of low-energy constants such as k_i is known from rather general considerations on effective theories [11] to be $k_i \simeq F_\pi^2/\Lambda^2$, where Λ is the typical mass of the massive states, not included in the effective theory, i.e. $\Lambda \simeq M_\rho$ (or M_K, M_η in the $SU(2) \times SU(2)$ expansion). This enables one to estimate that the corrective terms in (1) could be as large as $20 - 30\%$. Our claim is that by a clever use of τ-decay data recently released by the ALEPH collaboration [12, 15] it is actually possible to perform the sum rule evaluation of C in such a precise way as to actually provide an estimate for the combination of low-energy constants involved in the corrective terms in (1).

In practice, we advocate an approach in which one first constructs the QCD correlation function $\langle VV - AA\rangle$ in the chiral limit in *euclidian* space, an idea which was proposed in [16]. A key ingredient for this construction is the experimental measurement by the ALEPH collaboration [15] of the vacuum matrix elements of the dimension six and dimension eight combination of operators which control the first two terms in the asymptotic expansion of the chiral correlator. In euclidian space, far from the resonance region, this asymptotic expansion is expected to be accurate down to rather low momenta values, say $p \simeq 2$ GeV. The task is then to interpolate a smooth function of p, the value of which is known at zero (in terms of F_π in the chiral limit), in a finite momentum range. In this approach, the momentum integral in the $[0,\infty]$ range can then be performed exactly. It will be argued that the only knowledge of the two operator matrix elements (together with F_π) constrains the value of C to a level close to 10%. The estimate will then be refined by using more detailed experimental information on the vector and the axial-vector spectral functions.

2 Description of the method

The starting point is the sum rule derived by Das et al. [9] (a quick derivation can be found in [17]) written as an integral in four dimensional euclidian space. Performing the angular integration, one expresses the constant C as a one dimensional integral

$$C = \frac{3}{4}\frac{1}{16\pi^2}\int_0^\infty ds\, s\left[\mathring{\Pi}_A(-s) - \mathring{\Pi}_V(-s)\right] \quad (2)$$

where $\mathring{\Pi}_A$ and $\mathring{\Pi}_V$ are defined as the limit when $m_u = m_d = 0, e^2 = 0$ of the form-factors Π_A and Π_V associated with the axial-vector and the vector two-point correlation function. Π_V, for instance, is defined as

$$i\int d^4x\, e^{ipx}\,\langle 0|TV_\mu(x)V_\nu^\dagger(0)|0\rangle = (p_\mu p_\nu - p^2 g_{\mu\nu})\Pi_V(p^2) + p^2 g_{\mu\nu}\Pi_V^0(p^2)\ , \quad (3)$$

V_μ being the charged vector current $V_\mu(x) = \bar{u}(x)\gamma_\mu d(x)$. An exactly analogous definition holds for Π_A.

Formula (2) is exact provided chiral symmetry is spontaneously broken in QCD with two massless quarks. It is of interest to further consider the $SU(3)\times SU(3)$ chiral limit obtained by sending m_s to zero as well. However, as will be seen in the sequel, the uncertainties involved in this extrapolation are too large and do not permit a useful evaluation of $C_0 = \lim_{m_s=0} C$. Convergence of the integral in (2) follows from applying the operator-product expansion [18,19]. The operators must belong to the $(3,3)$ representation of the $SU(2)\times SU(2)$ group. In the limit $m_u = m_d = e^2 = 0$ the only such operators that one can construct are of dimension six or more. As these operators are order parameters for chiral symmetry transformations one a priori expects that their vacuum expectation values will be non-vanishing. The following asymptotic expansion therefore holds,

$$\lim_{p^2\to\infty} \mathring{\Pi}_A(-p^2) - \mathring{\Pi}_V(-p^2) = \frac{\lambda_6}{p^6} + \frac{\lambda_8}{p^8} + \dots \quad (4)$$

In QCD, λ_6 and λ_8 are not exactly constants except at leading order in α_s. At higher orders, corrections carry logarithmic-type p^2 dependences. In the leading-logarithmic approximation, this p^2 variation is found to be rather slow, such that the approximation of taking constant values for λ_6 and λ_8 will be accurate in a reasonably large energy region. We will return to this point in Sect. 4. The parameters λ_6 and λ_8 have been determined experimentally using τ decay data [15]. The method consists in using the analyticity properties of the two-point functions together with Cauchy theorem, which leads to equations like

$$\int_0^{M_\tau^2} ds P^{kl}(s)\left[\rho_A(s) - \rho_V(s)\right] = \frac{1}{2i\pi}\oint_{|z|=M_\tau^2} dz\, P^{kl}(z)\left[\Pi_A(z) - \Pi_V(z)\right] \quad (5)$$

where $P^{kl}(s)$ can be any polynomial. A convenient set is [20]

$$P^{kl}(s) = \left(1 - \frac{s}{M_\tau^2}\right)^{k+2}\left(\frac{s}{M_\tau^2}\right)^l\left(1 + \frac{2s}{M_\tau^2}\right)\ , \quad k,l > 0\ . \quad (6)$$

For $k = l = 0$ the left hand side of (5) reduces to a difference of total τ decay rates. These polynomials have the further merit to suppress the contributions which are close to the cut in the integral over the circle so that one can use asymptotic QCD expansions with some confidence in the righthand side of (5). Using this method, the ALEPH collaboration has determined the value of the following dimensionless integrals involving λ_6 and λ_8,

$$\delta^{(2n)} = -\frac{4\pi i}{M_\tau^2}\oint_{|z|=M_\tau^2} dz P^{00}(-z)\frac{\lambda_{2n}(z)}{z^n}, \quad n = 3,\ 4 \quad (7)$$

to be [15], $\delta^{(6)} = -0.058\pm0.006$ and $\delta^{(8)} = 0.0170\pm0.0014$. Ignoring the p^2 dependence of λ_6 and λ_8 (the validity of this approximation in the present situation can be checked explicitly for λ_6 and will be found in Sect. 4 to be excellent) one deduces,

$$\lambda_6 = (7.58\pm0.80)\,10^{-3}\ \mathrm{GeV}^6$$
$$\lambda_8 = (-1.07\pm0.12)\,10^{-2}\ \mathrm{GeV}^8\ . \quad (8)$$

The result for λ_6 is in reasonable agreement with that obtained earlier [21]. An analysis of the ALEPH data, making use of negative moments performed very recently [22] leads to values compatible with (8) although slightly smaller. It is perhaps important to stress that, even though λ_6 and λ_8 control the expansion of a chiral limit correlator they are effectively correctly determined from data in which $m_u, m_d \neq 0$. This is because quark mass effects are properly taken into account in the fit as they occur in the operator-product expansion via operators of lower dimensionality and the contribution of dimension six linear in the quark mass (involving the so-called mixed condensate) happens to vanish at leading order in α_s [23]. In other terms, the chiral correction to λ_6 is strongly suppressed. A priori, there is no reason for a similar suppression to hold for λ_8, but this parameter is of lesser practical importance in the calculation. A key assumption which is made in the above determination of λ_6 and λ_8 concerns, of course, the validity of truncating the asymptotic expansion (4) at order eight for $p = M_\tau$. We will see below that this assumption is internally consistent but it is not easy to estimate the error induced by this truncation. For this purpose, one should be able to determine more asymptotic parameters and check the stability of the determination.

Let us now explain the method for evaluating the integral in (2). We first split the integrand in two parts

$$\overset{\circ}{\Pi}_A(-s) - \overset{\circ}{\Pi}_V(-s) = \Pi^{exp}_{A-V}(-s) + \Pi^{rem}_{A-V}(-s) \ , \quad (9)$$

where $\Pi^{exp}_{A-V}(-s)$ is constructed from an experimentally measured part of the vector and the axial-vector spectral functions. We will proceed in three successive steps of approximation, including more and more experimental information in this part, and then check the stability of the result. In the first approximation, we include solely the pion pole part,

$$\Pi^{exp}_{A-V}(-s) = \frac{2F^2}{s} \quad (1^{\text{st}} \text{ approximation}) \quad (10)$$

where F is the pion decay constant $F_\pi \simeq 92.4$ (MeV) extrapolated to the chiral limit. The remainder part in (9), Π^{rem}_{A-V}, is reconstructed from its asymptotic expansion assuming that four terms in this expansion are known

$$\lim_{s\to\infty} \Pi^{rem}_{A-V}(-s) = \frac{a_2}{s} + \frac{a_4}{s^2} + \frac{a_6}{s^3} + \frac{a_8}{s^4} + \dots \ . \quad (11)$$

For instance, in the first order approximation corresponding to (10), one would have $a_2 = -2F^2$, $a_4 = 0$, $a_6 = \lambda_6$, $a_8 = \lambda_8$. The point is that, firstly, we expect this asymptotic expansion to become numerically accurate at rather low values of the momenta, $\sqrt{s} \simeq 2$ GeV. Secondly, the function $\Pi^{rem}_{A-V}(-s)$ is expected to be a perfectly smooth function down to $s = 0$. In the first order approximation, it has a logarithmic chiral singularity at $s = 0$ with a small numerical coefficient,

$$\lim_{s\to 0} \Pi^{rem}_{A-V}(-s) = \frac{1}{24\pi^2} \log s + cstt \quad (1^{\text{st}} \text{ approximation}) \ . \quad (12)$$

In higher order approximations this singularity will be exactly included in $\Pi^{exp}_{A-V}(-s)$ and the remainder part will be finite at $s = 0$. It is plausible that a simple rational approximation should be able to interpolate rather precisely the remainder function in the range $\sqrt{s} = [0, 2]$ GeV. One must restrict oneself to the class of diagonal Padé approximants in the variable $1/s$ in order to ensure finiteness at $s = 0$. The one of lowest degree which can match four terms in the asymptotic expansion gives

$$\Pi^{rem}_{A-V}(-s) = \frac{as + b}{s^2 + cs + d} \ . \quad (13)$$

The parameters of the approximant being related to those occuring in the asymptotic expansion (11) by the simple relations

$$d = \frac{a_6^2 - a_4 a_8}{a_4^2 - a_2 a_6}, \quad c = \frac{a_2 a_8 - a_4 a_6}{a_4^2 - a_2 a_6}, \\ b = a_4 + a_2 c, \quad a = a_2 \ . \quad (14)$$

In the second level of approximation we include into $\Pi^{exp}_{A-V}(-s)$ the most significant part of the 2π spectral function together with the one-pion pole which was considered before

$$\Pi^{exp}_{A-V}(-s) = \frac{2F^2}{s} - \int_0^{M_\tau^2} \frac{\overset{\circ}{\rho}_{2\pi}(x)}{x+s} dx \\ (2^{\text{nd}} \text{ approximation}) \ . \quad (15)$$

In this approximation, the logarithmic singularity (12) is properly taken into account provided the spectral function is correctly normalized at the origin: $\overset{\circ}{\rho}_{2\pi}(0) = 1/24\pi^2$ [10]. The construction of the chiral limit spectral function $\overset{\circ}{\rho}_{2\pi}$ knowing the experimentally measured one $\rho_{2\pi}$ is not a completely trivial matter and will be explained in the next section. The remainder piece is constructed as a Padé approximant as before except that the asymptotic expansion parameters a_i which enter are now given by

$$a_2 = -2F^2 + I_0 \ , \quad a_4 = -I_1 \ , \quad a_6 = \lambda_6 + I_2 \ , \\ a_8 = \lambda_8 - I_3 \quad (16)$$

with

$$I_n = \int_0^{M_\tau^2} dx \, x^n \, \overset{\circ}{\rho}_{2\pi}(x) \ . \quad (17)$$

One can of course think of continuing in this way and include more and more experimental information such that the remainder function will become numerically smaller together with the uncertainty associated with the Padé interpolation procedure. The next step, then, would be to include explicitly the contribution from the three pion component of the spectral function,

$$\Pi^{exp}_{A-V}(-s) = \frac{2F^2}{s} - \int_0^{M_\tau^2} \frac{\overset{\circ}{\rho}_{2\pi}(x)}{x+s} dx + \int_0^{M_\tau^2} \frac{\overset{\circ}{\rho}_{3\pi}(x)}{x+s} dx \\ (3^{\text{rd}} \text{ approximation}) \ . \quad (18)$$

What prevents one from pursuing this construction further lies in the difficulty of performing the chiral extrapolation, which increases with the pion multiplicity. It will fortunately appear that convergence is very fast, such that one hardly needs to go beyond the second approximation.

3 Chiral limit extrapolations

3.1 F_π

Extrapolation to the chiral limit of F_π can be performed fairly easily using known results from chiral perturbation theory. The value of F, corresponding to $m_u = m_d = 0$, $m_s \neq 0$ is related to F_π at one loop order by the following expression [10]

$$F = F_\pi \left(1 - \frac{13}{192\pi^2} \frac{M_\pi^2}{F_\pi^2} - \frac{M_\pi^2}{6} \langle r^2 \rangle_S^\pi + O(M_\pi^4) \right) \ , \quad (19)$$

which involves the scalar radius of the pion. Using for this quantity the updated value as given in [24]: $\langle r^2\rangle_S^\pi = 0.60 \pm 0.05$ fm^2, one obtains

$$F = 86.7 \pm 0.6 \pm 0.5 \text{ MeV} \tag{20}$$

using $F_\pi = 92.4 \pm 0.3$ MeV [25]. The second error in the value of F is a naive order of magnitude estimate of the size of the $O(M_\pi^4)$ correction in (19). This relatively precise extrapolation is to be contrasted with the situation in which one would be willing to further extrapolate to $m_s = 0$. Let F_0 be the corresponding limiting value of F_π, it is related to F by the following relation [26]

$$F_0 = F\left(1 - \frac{8m_sB}{F^2}L_4^r(m_sB) + O(m_s^2)\right) , \tag{21}$$

where B is proportional to the quark condensate in the chiral limit, $B = -\langle \bar{u}u\rangle/F^2$. This relation involves the low-energy constant L_4. Unfortunately, there is no independent way of determining L_4, which appears here multiplied by a large numerical factor.

3.2 $\rho_{2\pi}$

Let us now discuss the two-pion component of the vector spectral function. The possibility of performing a reliable extrapolation here is tied to the property, known for a long time, of vector meson dominance of the pion electromagnetic form-factor F_V. Defining $\rho_{2\pi}$ in terms of F_V,

$$\rho_{2\pi}(s) = \theta(s - 4M_\pi^2)\,\frac{1}{24\pi^2}\left(\frac{s-4M_\pi^2}{s}\right)^{\frac{3}{2}} |F_V(s)|^2 , \tag{22}$$

an excellent fit to the data can be performed up to the tau meson mass, with a Breit-Wigner function for the ρ resonance and only a small admixture of higher mass resonances,

$$F_V(s) = \frac{1}{1+\beta+\gamma}\left(B_\rho(s) + \beta B_{\rho'}(s) + \gamma B_{\rho''}(s)\right) , \tag{23}$$

with

$$B_\rho(s) = \frac{M_\rho^2}{M_\rho^2 - s - i\sqrt{s}\,\Gamma_\rho(s)},$$
$$\Gamma_\rho(s) = \theta(s-4M_\pi^2)\,\Gamma_\rho\,\frac{M_\rho^2}{s}\left(\frac{s-4M_\pi^2}{M_\rho^2-4M_\pi^2}\right)^{3/2} . \tag{24}$$

This type of parametrization guarantees that $F_V(0) = 1$ and was proposed in [27]. We will be using the numerical values obtained from a combined fit of the ALEPH $\tau \to 2\pi$ decay data and the $e^+e^- \to \pi^+\pi^-$ data [12]

$$\begin{aligned} &M_\rho = 773.4 \pm 0.9 \;\Gamma_\rho = 147.7 \pm 1.6 \;\beta = -0.229 \pm 0.020 \\ &M_{\rho'} = 1465 \pm 22 \;\Gamma_{\rho'} = 696 \pm 47 \;\gamma = 0.075 \pm 0.022 \\ &M_{\rho''} = 1760 \pm 31 \;\Gamma_{\rho''} = 215 \pm 86 . \end{aligned} \tag{25}$$

Other variants in the functional form of the Breit-Wigner function $B_\rho(s)$ may be used which would result in somewhat different values of the parameters (25). In particular, the form due to Gounaris and Sakurai [28] has better analytical properties and can approximately correctly reproduce the cut of $F_V(s)$ in the chiral limit while the simpler form (24) produces no cut at all. Nevertheless, for the problem at hand, we found numerically insignificant differences in using either parametrization.

It is clear that extrapolation to the chiral limit will dominantly affect the lower energy part of the spectral function. Furthermore, the uncertainties in the parameters of the higher resonances ρ', ρ'' are larger than the effect of setting $m_u = m_d = 0$. Therefore, in order to obtain the spectral function in the chiral limit it is only necessary to evaluate the extrapolated values of the ρ-meson mass and width, $\mathring{M}_\rho$ and $\mathring{\Gamma}_\rho$. Let us now discuss this issue.

In the case of the mass, firstly, one can perform a chiral expansion. At leading order, linear in the quark masses, the ρ and K^* masses are expressed in terms of two independent parameters (besides $\mathring{M}_\rho$) B_1 and B_8,

$$\begin{aligned} M_\rho &= \mathring{M}_\rho + 2\hat{m}B_8 + 2\hat{m}B_1 \\ M_{K^*} &= \mathring{M}_\rho + (m_s+\hat{m})B_8 + 2\hat{m}B_1 . \end{aligned} \tag{26}$$

One needs in principle to know both of these parameters in order to deduce $\mathring{M}_\rho$. The parameter B_8 is easily obtain $K^* - \rho$ mass difference,

$$2\hat{m}B_8 = \frac{2(M_{K^*} - M_\rho)}{r-1}, \quad r = \frac{m_s}{\hat{m}} \simeq 26 \tag{27}$$

(using the standard chiral expansion framework for evaluating the ratio $m_s/\hat{m}$). The value of B_1, on the other hand, cannot be simply determined, but this parameter is suppressed in the large N_c limit and thus should be smaller than B_8. Neglecting B_1 gives $\mathring{M}_\rho - M_\rho \simeq -10$ MeV. The chiral expansion of the vector meson masses has been pursued recently beyond linear order [29,30]. Including the leading correction, which are of order $O(m_q^{3/2})$ [29] gives for ρ-meson mass in the form,

$$\begin{aligned} M_\rho = &\mathring{M}_\rho + 2\hat{m}(B_8 + B_1) \\ &-\frac{g^2M_\pi^2}{48\pi F_\pi^2}\,\mathring{M}_K\,(3 + \frac{4}{3\sqrt{3}}) - \frac{g^2M_\pi^3}{12\pi F_\pi^2} . \end{aligned} \tag{28}$$

Here, the parameter g can be determined approximately to be $g \simeq 0.60$ [31,30] and $\mathring{M}_K = \lim_{m_u,m_d=0} M_K$. The parameter B_8 can, again, be determined from the $K^* - \rho$ mass difference and one finds that its numerical value is essentially the same as in the linear expansion. The corrective terms, even though suppressed in the large N_c limit, turn out to be relatively large and approximately cancel the contribution proportional to B_8. Further corrections of order $O(m_q^2)$ are also generated at one-loop which were computed in [30]. This contribution depends on a rather large number of parameters. We will not attempt to take it into account quantitatively but simply use the qualitative fact that it goes in the sense of reducing somewhat the large effect of the $O(m_q^{3/2})$ contribution, such that one can estimate with some confidence that the ρ mass in the chiral limit should lie in a range,

$$-10 \text{ MeV} \lesssim \overset{\circ}{M}_\rho - M_\rho \lesssim 0 \ . \tag{29}$$

Concerning the chiral limit of the width of the ρ-meson, we may also try to follow a similar approach and expand to linear order in the quark masses. Unfortunately, even at such a low order and dropping Zweig rule violating terms, there still remains too many undetermined constants. The most general chiral lagrangian terms describing vector meson coupling to pseudo-Goldstone boson pairs (using notations as in [33]) linear in the quark mass matrix are

$$\begin{aligned}\mathcal{L}_{vpp} = \frac{iG_V}{\sqrt{2}} \Big(&\langle V_{\mu\nu} u^\mu u^\nu \rangle + \gamma_1 \left\langle \{\chi^{(+)}, V_{\mu\nu}\} u^\mu u^\nu \right\rangle \\ &+ \gamma_2 \left\langle V_{\mu\nu} u^\mu \chi^{(+)} u^\nu \right\rangle \Big) \ . \end{aligned}\tag{30}$$

Note that wave-function renormalization effects of either the chiral fields or the vector meson fields can effectively be absorbed into the parameter γ_1. It turns out not to be possible to determine the three constants G_V (which determines the chiral limit width) and γ_1, γ_2 independently. Qualitatively, at least, this approach suggests, from the phase-space factor and the pion momentum dependence of the decay matrix element, that one should expect an increase of the ρ-meson width in the chiral limit of the order of 20%. This is a rather large effect and it must be properly taken into account.

As a way out of these difficulties, one may construct a set of sum rules involving the difference of the spectral functions $\rho_V - \overset{\circ}{\rho}_V$. To the extent that the lower part of the integration region dominates, such sum rules will efficiently constrain the chiral limit of the ρ-meson parameters. One derives a first sum rule by considering the combination of $\Pi_V(-s)$ minus its chiral limit counterpart $\overset{\circ}{\Pi}_V(-s)$. Asymptotically, one has (e.g. [34]),

$$\begin{aligned}&\lim_{s\to\infty} s \left(\Pi_V(-s) - \overset{\circ}{\Pi}_V(-s) \right) \\ &= \lim_{s\to\infty} \frac{-3}{8\pi^2} \Big\{ (1 + \frac{8\alpha_s(s)}{\pi})(m_u(s) + m_d(s))^2 \\ &\quad + (1 + \frac{2\alpha_s(s)}{\pi})(m_u(s) - m_d(s))^2 \Big\} = 0 \ . \end{aligned}\tag{31}$$

Hence, using a spectral representation, the following sum rule must hold,

$$\int_0^\infty dx \ (\rho_V(x) - \overset{\circ}{\rho}_V(x)) = 0 \ . \tag{32}$$

A second sum rule, with even better convergence properties, is obtained by considering the following $s = 0$ limit,

$$\Pi_V(0) - \lim_{s\to 0} \left(\overset{\circ}{\Pi}_V(s) + \frac{1}{24\pi^2} \log \frac{-s}{\mu^2} \right) \tag{33}$$

This expression can be evaluated in two different ways. Firstly, one can use the chiral expansion of the vector correlation function: a very good level of precision can be reached thanks to the calculation at two-loop order by Golowich and Kambor [35]. Secondly, one can write down a spectral representation: here it is convenient to split the integration range into $[0, 4M_\pi{}^2]$ and $[4M_\pi{}^2, \infty]$. In the first range, the integral can be performed explicitly, using the one-loop expression for the spectral function $\overset{\circ}{\rho}_V$. Equating these two evaluations, one derives the second sum rule,

$$\begin{aligned}&\int_{4M_\pi{}^2}^\infty dx \, \frac{\rho_V(x) - \overset{\circ}{\rho}_V(x)}{x} \\ &= \frac{1}{12\pi^2} \left(\log 2 - \frac{4}{3} \right) + \frac{M_\pi^2}{288\pi^4 F^2} \Bigg(\bar{l}_6 - \log 4 + \frac{8}{3} \\ &\quad + \frac{3}{2} (\bar{l}_5 - \bar{l}_6) \left[\log \frac{\mu^2}{M_\pi{}^2} + \frac{1}{4} \log \frac{\mu^2}{M_K^2} - \frac{1}{4} \right] \Bigg) \\ &\quad - \frac{8M_\pi^2}{F^2} \left[Q(\mu^2) + 2R(\mu^2) + \frac{F^2}{768\pi^2 M_K^2} \right] + O(M_\pi^4) \ . \end{aligned}\tag{34}$$

In this expression, $\bar{l}_5$ and $\bar{l}_6$ are low-energy constants which appear at $O(p^4)$ [10] and which are well determined, while $R(\mu^2)$ and $Q(\mu^2)$ are $O(p^6)$ constants [35] (the appearance of M_K in the above expression is related to the fact that these constants are appropriate for the three-flavour chiral expansion). One expects $R(\mu^2)$ to be suppressed compared to $Q(\mu^2)$ because of the Zweig rule (for values of the scale μ of the order of 1 GeV) and the latter constant was evaluated from a sum rule [36]

$$Q(M_\rho^2) = (3.7 \pm 2.0) 10^{-5} \ . \tag{35}$$

This enables one to evaluate the entire $O(M_\pi^2)$ contribution on the right-hand side of (34). The set of two sum rules (32) and (34) can be considered as a set of non linear equations from which one can determine $\overset{\circ}{M}_\rho$ and $\overset{\circ}{\Gamma}_\rho$. We have analyzed this system numerically, and found that it has a solution, which is unique in a physically meaningful range. Corresponding to the central values of the parameters cited above and including only the two-pion component of the vector spectral functions, one obtains

$$\overset{\circ}{M}_\rho - M_\rho = -2.4 \text{ MeV} \qquad \overset{\circ}{\Gamma}_\rho = 180.8 \text{ MeV} \ . \tag{36}$$

The uncertainties in this result come from two sources. Firstly, there is an uncertainty in the integrals of ρ_V coming from experimental errors in the parameters describing ρ_V. Varying the parameters in (25) one finds that, essentially, the error on M_ρ and Γ_ρ are the only ones that matter and that they translate into identical errors on $\overset{\circ}{M}_\rho$ and $\overset{\circ}{\Gamma}_\rho$. Secondly, we have neglected in the integrals the contribution of components in the vector spectral function other than 2π, i.e. $\rho_V^{4\pi}$, $\rho_V^{K\bar{K}}$, $\rho_V^{6\pi}, \ldots$ Evidently, one expects the first of the sum rules to be more sensitive to these contributions which set up at higher energies. One can make a rough estimate of the influence of these components using the quark-hadron duality idea, i.e. modelling the sum of all contributions by a continuum,

$$\rho_V^{cont}(s) = \frac{1}{4\pi^2}\theta(s - M_{cont}^2) \tag{37}$$

normalized to the asymptotic QCD prediction and starting at some threshold mass M_{cont}. A typical value used in sum rules analysis is $M_{cont} \simeq 1.5$ GeV. For the problem at hand, we need to know also how this continuum mass varies when going to the chiral limit. There is of course no way to precisely evaluate that, but it seems not unreasonable to assume $-10 \lesssim M_{cont} - \overset{\circ}{M}_{cont} \lesssim 10$ MeV, which leads to a variation $\Delta \overset{\circ}{M}_\rho = \pm 8$ MeV. The conclusion is that the chiral mass is, in fact, not determined to a better accuracy from the sum rules than it was from the chiral expansion as discussed above. Imposing that the sum rule result be the same as that found before, i.e. $\overset{\circ}{M}_\rho - M_\rho = -5 \pm 5$ MeV, is achieved by taking the continuum mass parameter in the range $\overset{\circ}{M}_{cont} - M_{cont} = 2 \pm 6$ MeV. Solving the two sum rule equations simultaneously yields the chiral mass and width as approximately linear functions of $\overset{\circ}{M}_{cont} - M_{cont}$ and they are found to lie in the range

$$\overset{\circ}{M}_\rho - M_\rho = -5 \pm 5 \text{ MeV} \qquad \overset{\circ}{\Gamma}_\rho = 180.0 \pm 1.5 \text{ MeV}\,. \tag{38}$$

One observes that the width gets determined with a much smaller error than the mass. The spectral function $\rho_{2\pi}$ and its chiral extrapolation are shown in Fig. 1. From this figure one observes, in particular, that the influence of setting m_u, m_d to zero is felt mostly in the low-energy region, $\sqrt{s} \leq 1$ GeV, consistently with the starting point assumption.

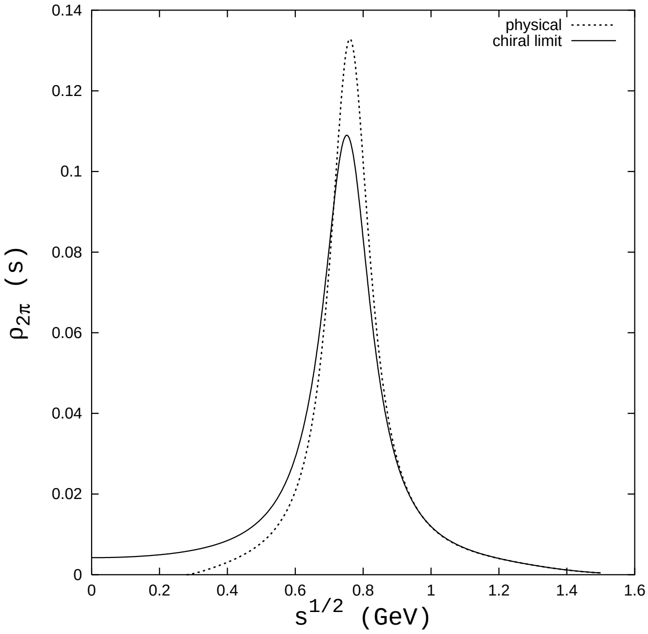

Fig. 1. Two-pion component of the vector spectral function and its chiral limit extrapolation obtained from solving the non-linear system of two sum rule equations, as explained in the text

3.3 $\rho_{3\pi}$

The spectral function piece $\rho_{3\pi}$ is not known to the same accuracy as $\rho_{2\pi}$. Furthermore, it will appear that extrapolation to $m_u = m_d = 0$ is plagued with larger uncertainties. However, $\overset{\circ}{\rho}_{3\pi}$ is not an essential ingredient, its explicit inclusion turns out to have very little effect and only serves to verify the stability of the calculation. For this purpose, an approximate knowledge of $\overset{\circ}{\rho}_{3\pi}$ may be sufficient. As before, one expects a sizable contribution from a resonance, the $a_1(1260)$ in this case. However, because the a_1 has a larger mass than the ρ and especially because it has a much larger width it is more questionable that the background contribution will be negligible. We will anyway follow the model of Kühn and Santamaria [27] which assumes complete dominance of the a_1 and matches with the correct chiral $O(p^2)$ behaviour of the axial current matrix element at low energy (note that the $O(p^4)$ expression has been recently worked out [37]). One assumption in this model is that the a_1 decays via a two step process: $a_1 \to \rho\pi \to 3\pi$ or $a_1 \to \rho'\pi \to 3\pi$ with a small probability. In principle, nothing prevents the a_1 decay to proceed also via the $a_1 \to \sigma\pi$ channel[1]. A clear signature for this process would be a difference in the a_1 decay rates into $2\pi^-\pi^+$ and $2\pi^0\pi^-$. These two rates have now been measured separately for the first time by the ALEPH collaboration [15] and found to be equal to a very good precision ($R^{--+} = 9.1 \pm 0.2\%,\ R^{00-} = 9.2 \pm 0.2\%$). This measurement supports the decay model of [27]. This model is embodied in the following parametrization

$$\begin{aligned}&\left\langle \pi^-(q_1)\pi^-(q_2)\pi^+(q_3)|\bar{u}\gamma^\mu\gamma^5 d|0\right\rangle \\ &\quad = -i\frac{2\sqrt{2}}{3F}B_{a_1}(s)\left(B_{\rho\rho'}(s_2)V_1^\mu + B_{\rho\rho'}(s_1)V_2^\mu\right)\end{aligned} \tag{39}$$

with

$$\begin{aligned}V_i^\mu &= q_i^\mu - q_3^\mu - Q^\mu\,\frac{Q.(q_i - q_3)}{s}, \\ Q &= q_1 + q_2 + q_3, \quad s_i = (Q - q_i)^2\end{aligned} \tag{40}$$

and

$$\begin{aligned}B_{\rho\rho'}(s_i) &= \frac{B_\rho(s_i) + \beta' B_{\rho'}(s_i)}{1+\beta'}, \\ B_{a_1}(s) &= \frac{M_{a_1}^2}{M_{a_1}^2 - s - iM_{a_1}\Gamma_{a_1}g(s)/g(M_{a_1}^2)}\end{aligned} \tag{41}$$

where $g(s)$ is a three-body phase-space integral which must be computed numerically (see [27] for more details [2]). We have determined the a_1 mass and width as well as the decay parameter β from a simple-minded fit of the ALEPH

[1] We approximate, as usual, a strongly interacting pion pair in an S-wave by a fictitious or real but very wide σ meson

[2] An approximate analytical form of $g(s)$ is given in this reference but one must be careful that it is only valid for the physical value of M_π and becomes incorrect for $M_\pi = 0$

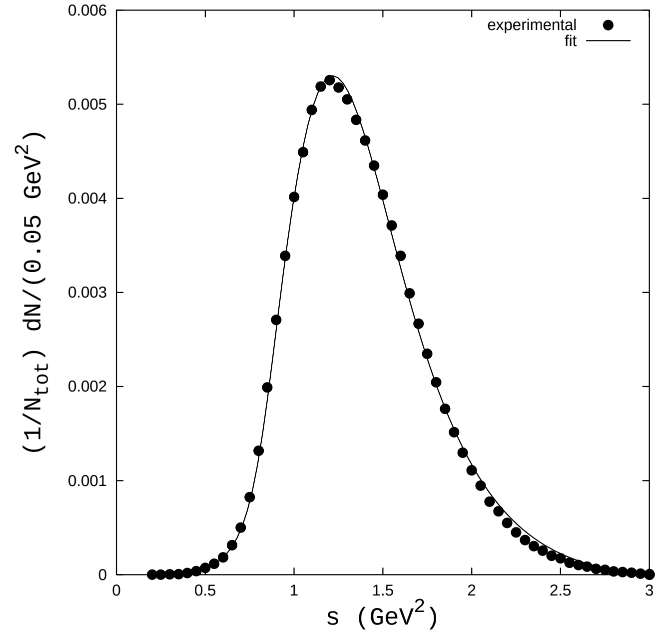

Fig. 2. Branching fraction for the mode $\tau \to \pi^-\pi^-\pi^+\nu$ as a function of the three-pion invariant mass squared. The experimental results from ALEPH are displayed together with our fit based on the Kühn-Santamaria parametrization

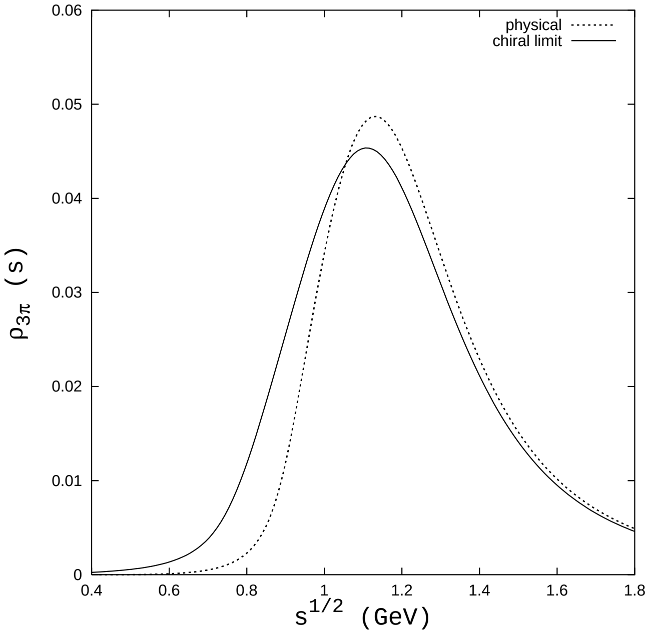

Fig. 3. Three-pion component of the axial-vector spectral function and its chiral limit extrapolation

data[3] [15] assuming energy independent errors. The resulting values for the a_1 parameters obtained in this way are,

$$M_{a_1} = 1.28 \pm 0.01 \text{ GeV}, \quad \Gamma_{a_1} = 0.67 \pm 0.05 \text{ GeV}, \\ \beta' = -0.27 \pm 0.03 \ . \qquad (42)$$

The experimental invariant mass distribution for the mode $\tau \to \pi^-\pi^-\pi^+\nu$ is shown in Fig. 2 together with the result of the fit using the above parametrization.

Now we would like to construct the chiral limit extrapolation of the 3π spectral function. As before, we disregard the modification of the parameters associated with the ρ' as it makes a relatively minor contribution to the spectral function. Concerning the ρ meson, the extrapolation of its mass and width were discussed in the previous subsection, there essentially remains to estimate the modification of the a_1 mass and width parameters. Concerning the mass, one encounters the first difficulty that the quark mass matrix not only shifts the 1^{++} multiplet but also mixes the states with non-zero strangeness with those of the 1^{+-} multiplet. Expanding to linear order in the quark masses, assuming ideal mixing, and using the $s\bar{s}$ member of the multiplet gives,

$$\overset{\circ}{M}_{a_1} \simeq M_{a_1} - \frac{M_{f_1(1510)} - M_{a_1}}{(r-1)} \ . \qquad (43)$$

This estimate must be considered as very approximate because of the additional problem that the assignment of the $f_1(1510)$ as the $s\bar{s}$ member of the a_1 nonet [25] is far from certain [38].

The value of the a_1 width, finally, in the chiral limit is constrained by a sum rule exactly analogous to (32) in the axial channel,

$$\int_0^\infty dx\ (\rho_A(x) - \overset{\circ}{\rho}_A(x)) = 2F^2 - 2F_\pi{}^2 \ . \qquad (44)$$

In this equation the one-pion component is excluded from ρ_A and its contribution appears on the right-hand side. As before, the additional assumption must be made that this equation constrains mostly the low energy part of the spectral function and, as a consequence, can essentially be interpreted as an equation for the 3π component of ρ_A. Using $\overset{\circ}{M}_{a_1} - M_{a_1} = -10$ MeV, the sum rule gives the chiral width: $\overset{\circ}{\Gamma}_{a_1} = 0.70$ GeV. We shall be content with a single sum rule here even though it is possible in principle to exploit a second sum rule in analogy with the case of the ρ meson. The result for the physical three-pion spectral function and its chiral limit is displayed in Fig. 3.

4 Results

Now that we have defined an approximation scheme for calculating Π^{exp}_{A-V} and Π^{rem}_{A-V} it is straightforward to compute the sum rule integral, (2). Before we do so, it is instructive to have a look at the integrand, which is displayed in Fig. 4 and Fig. 5 for the three levels of approximation.

Figure 4 shows the low energy region $0 \leq \sqrt{s} \leq 2$ GeV. One might believe that this part will dominate the integral, it actually turns out that the asymptotic tail makes

[3] The data can be found on the website http://alephwww.cern.ch/ALPUB/paper/paper98/1

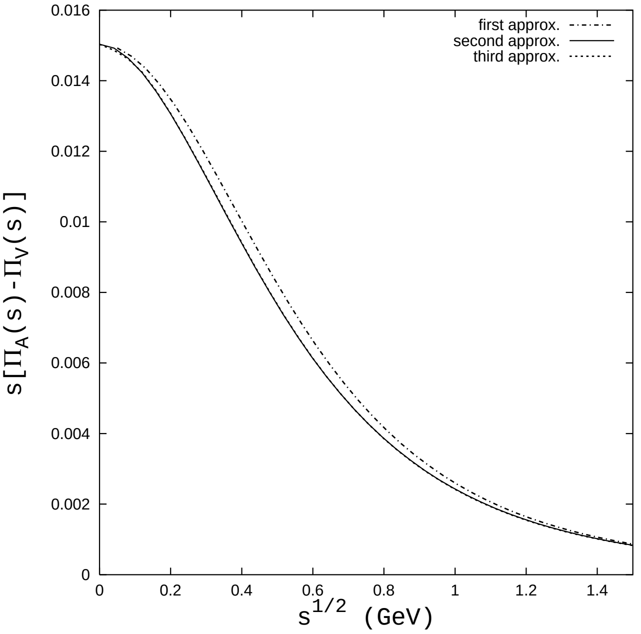

Fig. 4. Integrand of the Das et al. sum rule in the three successive approximations

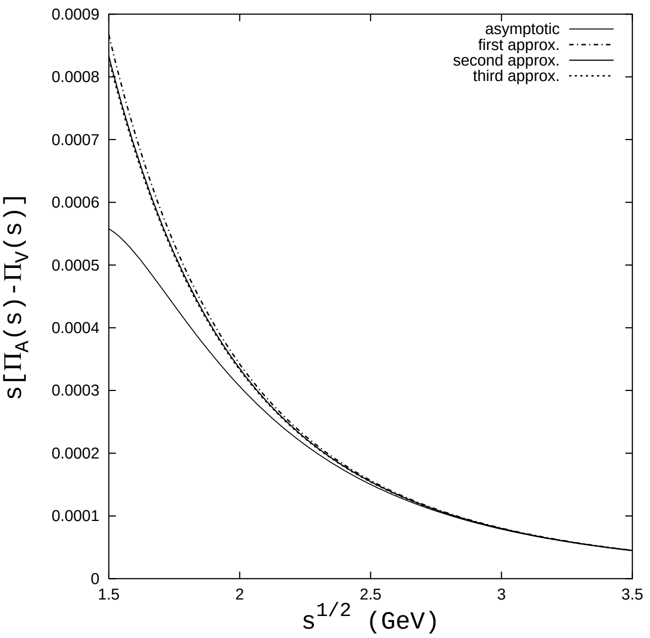

Fig. 5. Same as Fig. 4. Also shown for comparison is the two-terms asymptotic expansion of the integrand

a non negligible contribution of approximately 20%. It is one advantage of this method that it introduces no error due to truncation of the integral. One observes that approximations 2 and 3 generate curves which can hardly be distinguished. Figure 5 shows a region of larger values of the integration variable s from which one can appreciate the approach to the asymptotic regime. The two-terms asymptotic expansion is seen to be accurate at the 10% level for $\sqrt{s} = M_\tau$ and becomes very accurate provided

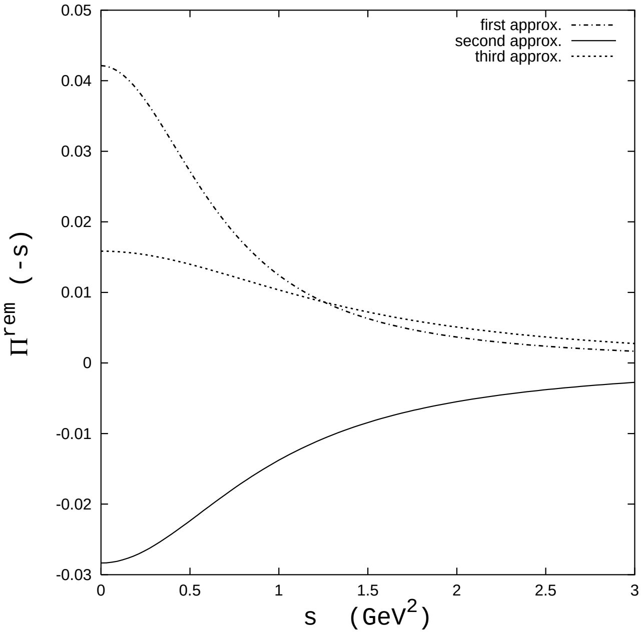

Fig. 6. Plot of the remainder part $\Pi^{rem}_{A-V}(-s)$ in the sum rule integrand (see Sect. 2) for the three successive approximations

Table 1. Numerical results from the sum rule (2) corresponding to the central values of the physical parameters. Z and δ are defined in the text

	Approx. 1	Approx. 2	Approx. 3
Z	0.899	0.854	0.852
δ	0.0177	0.0667	0.0683

$\sqrt{s} \gtrsim 2.5$ GeV. While the sum of Π^{exp}_{A-V} and Π^{rem}_{A-V} appear to be remarkably stable they are individually quite different from one approximation to the other. This is illustrated in Fig. 6 showing Π^{rem}_{A-V}, which is the part where the Padé interpolation procedure is used: the figure shows how this part becomes smaller as one includes more experimental information from the spectral functions. The curves are seen to be smooth, flat, an exhibit no change of sign thereby justifying, a posteriori, the use of a simple rational approximation.

We can now perform a stringent test of the interpolation procedure by considering the integrand at low energy, comparing it with the chiral perturbation theory expectation,

$$\lim_{s\to 0}\left[-24\pi^2\left(\mathring{\Pi}_{A-V}(-s) - \frac{2F^2}{s}\right) + \log\frac{s}{{M_\pi}^2} - \frac{5}{3}\right] = \bar{l}_5 \ . \tag{45}$$

The low-energy constant $\bar{l}_5$ is known from the one-loop analysis of the charge radius of the pion $\langle r^2\rangle^\pi_V$ and of the pion radiative decay amplitude, $\pi \to e\nu\gamma$: $\bar{l}_5 = 13.1 \pm 1.3$ (using $F_\pi = 92.4$). An additional $5 - 10\%$ uncertainty is expected from $O(M_\pi^4)$ contributions to these observables, which necessitate a two-loop analysis (at present only $\pi \to e\nu\gamma$ has been analyzed at this level of accuracy

[39]). Using our construction for $\Pi_{A-V}(-s)$ and computing numerically the limit (45) we find,

$$\bar{l}_5 = 11.8 \tag{46}$$

(the result differ in approx.2 and approx.3 by less than 1%) which is slightly smaller but compatible with the one-loop determination quoted above. This is a rather non trivial check of the quality of the interpolation from the low-energy domain of the chiral expansion up to the domain of large energies, where the operator-product expansion makes sense. The result of the sum rule evaluation of the low-energy constant C are displayed in Table 1. We show firstly the dimensionless quantity $Z = C/F^4$ which is of order unity and also the quantity $\delta \equiv (M^2_{\pi^+} - M^2_{\pi^0} - 2e^2C/F^2)/(M^2_{\pi^+} - M^2_{\pi^0})$ which measures the relative importance of the subleading terms in the expansion of the $\pi^+ - \pi^0$ mass difference.

The contribution of the subleading terms is predicted to be positive and have a relative magnitude of 7%. This, of course, is in agreement with the upper bound that one obtains from naive dimensional analysis of the low energy constants, which is 20%. What is the accuracy of this evaluation? We can identify three sources of error: 1) the error coming from the uncertainties in the physical parameters that enter the calculation. 2) An error coming from the chiral limit extrapolation and 3)an error associated with the assumption that λ_6 and λ_8 are constants, which is only an approximation. Concerning the first source of error, we have varied all the physical parameters independently and calculated the variation of the result for both approximations 2 and 3. The result is shown in Table 2 below. The parameters which are not shown like $M_{\rho'}$, $\Gamma_{\rho'}$ induce very small errors. It is interesting that the individual errors are rather different in the two approximations. For instance, the error induced by F (here the chiral limit extrapolation error was included as well) or by λ_6, λ_8 are significantly smaller in approx. 3 than in approx. 2. This does not imply that the third approximation has a smaller error, as it exhibits a greater sensitivity to the tail of the vector spectral function. If one simply adds all the errors one finds very closely the same number for the two approximations, respectively 7.3% and 7.5%. This is suggestive that both the central value and the error are just as reliably obtained from approx.2. In this approximation, we can also estimate the error due to the evaluation of the chiral limit values $\overset{\circ}{M}_\rho$ and $\overset{\circ}{\Gamma}_\rho$. Varying the continuuum contribution in the set of sum rules as discussed in Sect. 3.2 we obtain a small contribution of 0.2%.

The last uncertainty arises from the assumption made so far that λ_6 and λ_8 are constants which is only true at leading order in α_s. This point can be investigated quantitatively in the case of λ_6. Using the anomalous-dimension matrix provided in [23], one can resum the leading logarithms and obtain,

$$\begin{aligned}
&\lambda_6(s) \\
&= \frac{64\pi\alpha_s(\mu)}{9}\Bigg\{(O^a_6(\mu) + \frac{1}{6}O^b_6(\mu))\left[1 + \frac{9\alpha_s(\mu)}{4\pi}\log\frac{s}{\mu^2}\right]^{-1/9} \\
&\quad +(\frac{1}{8}O^a_6(\mu) - \frac{1}{6}O^b_6(\mu))\left[1 + \frac{9\alpha_s(\mu)}{4\pi}\log\frac{s}{\mu^2}\right]^{-10/9} \\
&\quad +\frac{9\alpha_s(\mu)}{32\pi}\left[\frac{119}{6}O^a_6(\mu) + O^b_6(\mu)\right]\Bigg\} .
\end{aligned} \tag{47}$$

In this expression, O^a_6 and O^b_6 are the vacuum expectation values of the two operators

$$\begin{aligned}
O^a_6 &= \left\langle \bar{u}\gamma_\mu\gamma^5\frac{\lambda^a}{2}d\bar{d}\gamma^\mu\gamma^5\frac{\lambda^a}{2}u - \bar{u}\gamma_\mu\frac{\lambda^a}{2}d\bar{d}\gamma^\mu\frac{\lambda^a}{2}u\right\rangle \\
O^b_6 &= \langle \bar{u}\gamma_\mu\gamma^5 d\bar{d}\gamma^\mu\gamma^5 u - \bar{u}\gamma_\mu d\bar{d}\gamma^\mu u\rangle
\end{aligned} \tag{48}$$

with λ^a a color-space Gell-Mann matrix. In principle, in order to take the s-dependence correctly into account, one needs to know the values of both O^a_6 and O^b_6. However,these two operators are not exactly on the same footing since O^b_6 appears multiplied by one factor of α_s more than O^a_6 whenever the logarithm is not too large. Furthermore, O^b_6 is suppressed in the large N_c limit. A plausible approximation, then, would be to ignore it altogether. Another plausible approximation is that of vacuum saturation [23], which yields the relation,

$$O^b_6 = \frac{3}{4}O^a_6 . \tag{49}$$

The energy dependence of λ_6 is very much suppressed in this approximation. As far as the integration over the circle in the complex plane is concerned (see (7)), we find that dropping the energy dependence is a very good approximation in any case, which does not generate an uncertainty in the determination of $\lambda_6(M_\tau)$ larger than 1%. One observes from (47) that λ_6 is a steadily decreasing function of s. Our construction can be seen as a procedure for smoothly matching the low-to-medium and the high energy regimes [16]. In this sense it is clear that one must choose $\lambda_6 \equiv \lambda_6(s_0)$ where s_0 is the value of s where the asymptotic regime sets in, i.e. s_0 must lie between M^2_τ and $2M^2_\tau$, say, as can be seen from Fig. 4. This determines the constant value of λ_6 to use within 2% approximately. Then, one must take into account the contribution of the logarithms in the high energy region of the sum rule integral. This is found to introduce a rather small correction to the value of C which ranges from 0.4 to 0.8% depending on the hypothesis made for O^b_6. In conclusion, we obtain that the overall relative error in the determination of the parameter C does not exceed 10%.

Let us now consider the implication of this result for $O(p^4)$ low-energy parameters using the chiral expansion of the $\pi^+ - \pi^0$ mass difference at this order [7],

$$\begin{aligned}
&M^2_{\pi^+} - M^2_{\pi^0} \\
&\quad = \frac{2e^2C}{F^2}\left(1 - \frac{M^2_\pi}{16\pi^2F^2}\left(3\log\frac{M^2_\pi}{\mu^2} + 1\right)\right) \\
&\qquad +\frac{e^2M^2_\pi}{16\pi^2}\left(-3\log\frac{M^2_\pi}{\mu^2} + 4\right) + \frac{2M^4_\pi}{F^2}\left(\frac{m_d - m_u}{m_d + m_u}\right)^2 l_7 \\
&\qquad +2e^2M^2_\pi F_k(\mu) + O(e^4)
\end{aligned} \tag{50}$$

Table 2. Percentage relative variation of the result for C/F^2 corresponding to the variation of the various input physical parameters within their error bars. The second and third lines of the table correspond to the calculation in approximation 2 and 3 respectively

parameters	F	λ_6	λ_8	M_ρ	Γ_ρ	β	γ	M_{a_1}	Γ_{a_1}
error(2)	0.9	5.1	1.0	0.02	0.04	0.2	0.05	–	–
error(3)	0.04	3.4	0.2	0.01	0.4	1.4	1.5	0.2	0.4

with

$$F_k(\mu) = -2k_3^r(\mu) + k_4^r(\mu) + 4k_6^r(\mu) + 4k_8^r(\mu)\ . \qquad (51)$$

The $O(e^4)$ contribution which must technically be counted as $O(p^4)$ can be estimated to be numerically smaller by one order of magnitude than the $O(e^2M_\pi^2)$ or the $O((m_u-m_d)^2)$ ones and is neglected here. From this, and using $m_u/m_d = 0.55$ [40], one deduces,

$$2.2\, l_7 + F_k(M_\rho) = (-7.1 \pm 3.0)\, 10^{-2} \qquad (52)$$

which is our main result. For comparison, on the basis of naive dimensional analysis alone, one would obtain for the same quantity that it must lie in the range $\pm 8\, 10^{-2}$. The parameter l_7 which appears in (52) is not very precisely known but a simple resonance-saturated sum rule gives an estimate [10] $l_7 \simeq 0.7\, 10^{-2}$.

5 Conclusion

To summarize, we have attempted an evaluation of the low-energy constant C with a controlled error, on the basis of the exact sum rule expression of Das et al.. The main practical difficulty, which is present even if infinitely precise experimental data were available, lies in the necessity of extrapolating the integrand to the chiral limit. A calculational procedure was proposed in which one first reconstructs the relevant current-current correlator in euclidian space making use of its smoothness properties together with the experimental determination of two asymptotic expansion parameters. An approximation scheme can be developped in which one includes spectral function components with higher and higher pion multiplicities. This expansion was argued to converge very rapidly such that, in practice, it is only necessary to include the one-pion and the two-pion components. The construction of the chiral limit makes use of recent work both on application of chiral perturbation theory to the vector meson masses and of chiral calculations at two-loop order of current-current correlation functions. We have shown that under the assumption that the relative error on the asymptotic parameter λ_6 is of the order of 10% (this is the actual experimental error but it does not include the uncertainty stemming from the truncation of the OPE, which is more difficult to evaluate), one can determine the parameter C with an error of slightly less than 10% and deduce a meaningful estimate for a combination of subleading parameters k_i. These parameters are primarily useful in calculations of radiative corrections at low energy. Another area where the computation of the photon loop is of interest, is in relation with the $K^+ - K^0$ mass difference and the issue of Dashen's theorem violation. It is possible that the constraint obtained here may prove useful in this context as well.

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